Lobachevsky and Euclid

1. In one sense, both systems are simply logical systems, thought experiments.
2. What is the point of a logical system? It seems to be to develop a small set of propositions that in combination lead to secondary propositions which are more complex, less obvious, and yet necessarily true, or rather, non-contradictory to the original propositions and to all other propositions in the same system.
3. Yet, with regard to geometry, we have an urge to map these propositions onto the real world; we have a sense that they are describing the real world, and evaluate their "truth" on the basis whether the real world behaves like the system.
4. If the issue were simply logical cohesiveness with interesting implications, the 5th postulate should not be worrisome; it would be simply a rule of that particular system.
5. As soon as we say, "I'm not sure about the 5th postulate", we reveal that we are evaluating the truth of this postulate in light of something else. What?
6. Perhaps we are saying we wonder whether this postulate is the only one which can produce interesting and consistent results. The answer is obviously "no" as soon as another assumption is seen to produce such results, such as Lobachevsky's.
7. Perhaps the interest is in certain knowledge, the kind that math provides. Yet the knowledge is only as certain as the original group of axioms.
8. Euclid's geometry is developed as a special limiting case of Lobachevsky's.
9. It is quite startling that Lobachevsky's geometry provides an absolute distance.
10. It seems to me that it also means that space is not scalable, in the sense used in Poincare's essay.
11. Yet, Lobachevsky's geometry does not give a p for any particular angle, so we cannot ascertain whether his geometry describes the real world. Any measurement, no matter how fine, could be argued to be simply a too-small triange.
12. In one sense, special relativity confirms Euclid, in that thought experiments using light-clocks, which assume the Pythagorean theorem, give the correct formula for time dilation. In another sense, is invalidates Euclid, as in the spinning disc.
13. What are we after here?
14. Why is angle absolute in Euclid? Angle has to do with two dimensions, but is relative. Length has to do with only 1, and is also relative. What is the 3 dimensional analog?

15. When considering geometry, and its fundamental concepts, there is a sense of being confined to some set of fundamental ideas, of which others are either inconceivable or simply composites.

Euclid, Lobachevsky, and Space

A Reflective Essay

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What, exactly, are we doing when we think about geometry? Are we simply elaborating a beautiful pattern and then appreciating our handiwork? Are we exploring and describing the deep structure of our world? Are we simply the kind of being that cannot help inventing languages and symbolic systems?

At one level, both Euclid and Lobachevsky have presented logical systems which are a pleasure to explore, and which exercise the mind in the same manner as a good dimensional or logical puzzle. In fact, part of the appeal of the activity of studying geometry may be the coincidence, in one activity, of logical, spatial, algebraic and linguistic challenges. Both the left and the right brain are exercised. We are diverted.

I believe our attraction to these logical systems stems also from the certainty of knowledge they appear to provide. In the contemplation of these systems we find refuge from the equivocation and compromise of daily life, from the cacophony of differing opinion that we experience if we are in any way in touch with our society. These systems start with a mere handful of ideas, chosen to be as self-evident as possible, and proceed to elaborate every subsequent idea in a deductive manner, so that our certainty about any of them is as deep as our certainty about the initial ideas. At any stage of the endeavor, if the proofs have been well done, we can enjoy the feeling of knowing something for certain.

Of course, the certainty can never exceed that with which we hold the initial ideas, the axioms and postulates. Furthermore, we know that any uncertainty at all in these foundational ideas is likely to be compounded in any recursive activity, especially if that activity is linear, or chain-like. Therefore, we feel more certain if the system is web-like, wherein key ideas can be developed along different logical pathways that converge, as it were, on nodes of multiply-consistent derivative ideas. In such a conceptual web, one could perhaps begin with any two or three complex theorems some way into the geometry, assume their truth, and then work backwards to demonstrate the initial axioms as true. In this way we would find, for example, that the assumption of the Phythagorean theorem as true might enable us to prove that Euclid's fifth postulate, or some derivative of it (like the Neal Postulate) is true.

But what do we mean by this certainty? If the objective is to develop an elaborate, self-consistent pattern of statements that offers beauty and complexity to the mind's eye, what sense does it make to inquire whether the axioms are certain? One could answer that part of the game is to use as few initial statements as possible, and still be able to derive a beautiful, complex system. Accordingly, a geometry that omitted Euclid's fifth postulate would be more interesting, and one that also omitted his fourth postulate still more so. In this case, "I am uncertain about Euclid's fifth postulate" would be taken to mean, "I am not certain that this postulate is necessary to create an interesting geometry." It would not be uncertainty about its "truth", but only about its necessity to the aesthetics of thought.

Alternatively, the certainty may pertain to whether the thinker can "perceive very clearly" (Descartes' phrase) that the postulate is "true" in that internal world of interrelated concepts that we each seem to create, or find, in our minds. The exact nature of this world is difficult to know and describe, as it can only be directly contemplated in one's own thoughts, and it is doubtful whether there is any uniformity between one person's internal world and another's. It seems likely to me that this internal world is the representation, to our consciousness, of our foundational beliefs about the external, *re*-al world, as continually modified by our perceptions and by the categories available to us through language. The structure of this world seems to be plastic. The experience of "seeing" some idea as true, rather than simply granting it, may correspond to the discovery that it fits comfortably in this internal world, or that by "installing" the idea in this world, the whole construct appears more coherent to the mind's eye.

We are apparently able to compartmentalize this inner world somewhat, in that we can conceive of alternate worlds, and have a sense whether a given idea would fit into that alternate world. A thinker might maintain several alternate worlds in his mind, one of which he believes to represent the real world.

Euclid's elements provide a three-dimensional world in which space is uniform and infinitely extensible, parallel lines do not approach each other, and triangles contain two right angles. For most of us, for most of history, this is the world we believed to represent the real world.

Lobachevsky is uncertain whether there is only one non-cutting line through a point off another line. In his Theorem 16 he begins to develop an alternate world, based on the single concept that there may be more than one non-cutting line to a given line through a point off that line. In Theorem 22 he demonstrates that in this world the sum of the angles within any and all triangles is smaller than two right angles. In Theorem 24 he shows that parallel lines must continually approach each other, yet never meet, in the direction of their parallelism. Theorem 23 demonstrates that for every angle there is a distance from a given line at which that angle is the angle of parallelism, with a corollary that for every distance there is only one angle of parallelism.

This is a very odd world. There are no such things as squares or rectangles. There can be congruent triangles, but no similar triangles. An equilateral triangle's angles will diminish as its size increases, so that at some size it will have each angle only 1 degree! Not only do parallel lines not stay the same distance apart, but also no two lines anywhere can stay the same distance apart. Even though the mind "sees" such lines as curved, yet one can rotate them about any two of their points and they still lie on themselves.

To me, the most intriguing aspect of Lobachevsky's world is that it is non-scalable; it implies a certain absolute distance. Poincar? has pointed out that, if the world were overnight to double its every dimension, we would be absolutely unable to detect that anything had happened. (Science and Method, 1897) This seems to me to presume a Euclidean world in which similar triangles are possible. Lobachevsky's Theorem 23 establishes a functional relationship between angle and distance, in which there is only one specific distance p at which a given angle is the angle of parallelism. Hence, if the world were scaled up or down, previously parallel lines would no longer be so, certain triangles would “open up” if grown and certain parallel lines would meet as angles if shrunk. This would play havoc upon architecture, to say the least!

This also suggests that Lobachevsky's world provides a method for testing its correspondence to the real world. If, in fact, the space in this real world is like the space in Lobachevsky's world, then one should be able to find a particular distance and particular angle which satisfy the parallel-angle function in this particular world. We should be able to measure a decreasing sum of angles in triangles as they are scaled up in size. In actual practice, we would have the difficulty of not knowing how big a triangle we would need in order to detect a defect in the sum of the interior angles, but the detection is theoretically possible. In the event that such a measurement were made and discovered to support Lobachevsky's world, it should cause anyone who can comprehend such an alternate world to identify it as the real world in his mind's eye.

Yet, the establishment of a single geometry as corresponding to real space seems unlikely to me, in light of the theories of special and general relativity. Though both these theories generate curved space, the curvatures are not uniform over all space but are peculiarly local and dependent upon matter-energy structures in any given vicinity. In a room with a spinning disc, each physical item distorts space ever so slightly because of its mass, and the space that the spinning disc occupies is distorted radially by the smoothly varying speeds of each of its cylindrical parts. This almost hopeless non-uniformity of space, not to mention the doubt whether space as such even exists apart from the things extended in it, makes it always a thing apart from our internal models.